Differentiable rings, analytic rings, Nash rings and their applications to singularity theory (Singularity theory of differential maps and its applications)

Yamashita, Tatsuya

数理解析研究所講究録
第2049巻 2017年 22-31

Differentiable rings, analytic rings, Nash rings and their applications to singularity theory

Tatsuya Yamashita
Department of Mathematics,
Hokkaido University

1 Introduction

This is a survey paper which was presented by the author in the RIMS at Kyoto University. This paper contains several recent results which obtained by the author. Let us mention on the motivations of our study related to the theory of manifolds and that of C^{∞}-rings.

Let M, N be C^{∞}-manifolds and $C^{\infty}(M)$ (resp. $C^{\infty}(N)$) a set of C^{∞}-functions on M(resp. N). $\mathfrak{C}=C^{\infty}(M)$ is a kind of "C^{∞}**-ring**" with the following property: for any $l\in\mathbb{N}$ and $f\in C^{\infty}(\mathbb{R}^{l})$, there exists an operation $\Phi_{f}:\mathfrak{C}^{l}\rightarrow\mathfrak{C}$ defined as $(\Phi_{f}(h_{1},\ldots,h_{n}))(x):=f(h_{1}(x),\ldots,h_{n}(x))$ for $x\in M$ for $h_{1},\ldots,h_{l}\in\mathfrak{C}$ (Definition 2.1. [5]). For an analytic manifold (resp. a Nash manifold) M, we can define an analytic-ring $C^{\omega}(M)$ (resp. a Nash-ring $\mathcal{N}^{\omega}(M)$).

For a C^{∞}-map $f:M\rightarrow N$ of C^{∞}-manifolds, there exists a pullback $f^{*}:C^{\infty}(N)\rightarrow C^{\infty}(M)$ defined as $f^{*}(c):=c\circ f\in C^{\infty}(M)$ for $c\in C^{\infty}(N)$.

We can regard a C^{∞}-tangent vector field $V:M\rightarrow f^{*}(TN)$ over f on M as an $\mathbb{R}$**-derivation** $V:C^{\infty}(N)\rightarrow C^{\infty}(M)$ along f^{*}, i.e. V is an $\mathbb{R}$-linear map with the following property

$$V(h_{1}h_{2})=f^{*}(h_{1})V(h_{2})+f^{*}(h_{2})V(h_{1}) \text{ for all } h_{1},h_{2}\in C^{\infty}(N).$$

Note that in this case, V turns to be a C^{∞}**-derivation**, i.e. V satisfies that:

$$V(g\circ(h_{1},\ldots,h_{l}))=\sum_{i=1}^{l}\left(\frac{\partial g}{\partial x_{i}}\circ(f^{*}(h_{1}),\ldots,f^{*}(h_{l}))\right)\cdot V(h_{i}),$$
$$\text{for any } l\in\mathbb{N},\ h_{1},\ldots,h_{l}\in C^{\infty}(N), \text{and } g\in C^{\infty}(\mathbb{R}^{l}).$$

Therefore, any $\mathbb{R}$-derivation $V:C^{\infty}(N)\rightarrow C^{\infty}(M)$ along f^{*} is a C^{∞}-derivation.

Let $\mathfrak{C}$ be a C^{∞}-ring and $\mathfrak{M}$ be a $\mathfrak{C}$-module with a $\mathfrak{C}$-homomorphism $\phi:\mathfrak{C}\rightarrow\mathfrak{M}$. When (under which condition of $\mathfrak{C},\mathfrak{M},\phi$) does an $\mathbb{R}$-derivation $V:\mathfrak{C}\rightarrow\mathfrak{M}$ over ϕ become a C^{∞}-derivation? In [9] for a C^{∞}-ring $\mathfrak{C}$ and a C^{∞}-ring $\mathfrak{M}=\mathfrak{D}$ regarded as a $\mathfrak{C}$-module, any $\mathbb{R}$-derivation V is a C^{∞}-derivation if $\mathfrak{D}$ is k-jet determined. In [4] for a Nash-function $f:\mathbb{R}^{n}\rightarrow\mathbb{R}$, $V(\Phi_{f}(c_{1},\ldots,c_{n}))-\sum_{i=1}^{n}\Phi_{\frac{\partial f}{\partial x_{i}}}(c_{1},\ldots,c_{n})V(c_{i})$ become a nilpotent element of $\mathfrak{M}$ for any $c_{1},\ldots,c_{n}\in\mathfrak{C}$.

In Definition 4.16. [5], for a category $\mathbf{C^{\infty}Rings}$ of C^{∞}-rings and a category $\mathbf{LC^{\infty}RS}$ of local C^{∞}-ringed spaces, there exists a functor Spec : $\mathbf{C^{\infty}Rings}^{\mathrm{op}}\rightarrow\mathbf{LC^{\infty}RS}$. A C^{∞}-manifold M is regarded as a "C^{∞}**-scheme**" $Spec(C^{\infty}(M))$. Therefore, we can regard a C^{∞}-manifold M as a "space associated with $C^{\infty}(M)$" and a tangent vector field over M as a "derivation $C^{\infty}(M)\rightarrow C^{\infty}(M)$". For analytic-rings, Nash-rings, we try to regard analytic-manifolds and Nash-manifolds as "space associated with analytic-rings and Nash-rings". To define and study of singular point and vector fields on $\mathcal{K}$-schemes for $\mathcal{K}=C^{\infty},C^{\omega},\mathcal{N}^{\omega}$, we study properties of derivations $V:\mathfrak{C}\rightarrow\mathfrak{C}$ of $\mathcal{K}$-rings.

In §2, we recall the notions of C^{∞}-functions, analytic-functions, and Nash-functions. These functions are closed under sums, products, and partial derivations $\frac{\partial}{\partial x_{i}}$.

In §3, we recall the notions of C^{∞}-rings, $\mathbb{R}$-derivations and C^{∞}-derivations. Then we define analytic-rings, Nash-rings and their derivations. We can define ideals of C^{∞}-rings (resp. analytic-rings, Nash-rings) and their quotient C^{∞}-rings (resp. analytic-rings, Nash-rings).

In §4, we show the properties for C^∞-rings from the properties of C^∞-functions and its germs. First, we have a localization of $C^\infty(M)$ at p as a set $C^\infty_p(M)$ of germs of C^∞-functions at p. Second, we compare the difference of $\mathbb{R}$-derivations and C^∞-derivations of C^∞-rings.

In §5, we show the properties for analytic-rings from the properties of analytic-functions and its germs. we have that localizations of $C^\omega(M)$ at p is not isomorphic to $C^\omega_p(M)$.

In §6, we show the properties for Nash-rings. From [4], we introduce the properties of Nash-derivations, i.e. $\mathbb{R}$-derivations which satisfies Leibniz rule for Nash-functions.

2 The kinds of functions

2.1 Definition of functions

For $\alpha = (\alpha_1, \ldots, \alpha_n)(\alpha_i \in \{0\} \cup \mathbb{N})$, define $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\alpha! := \prod_{i=1}^n \alpha_i!$.

Definition 1 1. *A "C^∞-**function(A smooth function)**" $f : \mathbb{R}^n \to \mathbb{R}$ is a function which satisfies: for any positive integer $r \geq 0$ and $\alpha \in (\{0\} \cup \mathbb{N})^n$ with $|\alpha| \leq r$, there exists the r-th partial derivative $\frac{\partial^\alpha f}{\partial x^\alpha} := \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} : \mathbb{R}^n \to \mathbb{R}$ which is continuous on $\mathbb{R}^n$.*

2. *A "C^ω-**function(An analytic function)**" $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞-function which satisfies: for any $p \in \mathbb{R}^n$, there exists an open neighborhood U such that $\sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(p)(x-p)^\alpha : U \to \mathbb{R}$ formally converges to f on U.*

3. *For $r = 0, 1, 2, \cdots, \infty, \omega$, a "$\mathcal{N}^r$-**function(a C^r Nash function)**" $f : \mathbb{R}^n \to \mathbb{R}$ is a C^r-function which satisfies: there exists a non-zero real polynomial $P(x, y) \in \mathbb{R}[x, y]$ such that $P(x, f(x)) = 0$ on $\mathbb{R}^n$ ([8]). We call C^ω-Nash functions as **Nash-functions**.*

There exists a C^∞-function (resp. a C^ω-function) which is not the C^ω-function (resp. the $\mathcal{N}^\omega$-function) from following examples.

Example 1 1. *Let η be a function defined as $\eta(x) = \begin{cases} e^{-\frac{1}{x}} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$. η is the C^∞-function but not the analytic function. The derivation of η is $\eta'(x) = \begin{cases} \frac{1}{x^2} e^{-\frac{1}{x}} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$ and continuous. Then, the r-th derivation $\eta^{(r)}$ is*
$\eta^{(r)}(x) = \begin{cases} P_r(\frac{1}{x}) e^{-\frac{1}{x}} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$ *for a real polynomial* $P_r(x) = \begin{cases} x^2 & (r = 1) \\ -x^2 P'_{r-1}(x) + x^2 P_{r-1}(x) & (r \geq 2) \end{cases}$.
$\eta^{(r)}$ is continuous for any $r \geq 0$. Therefore, η is a C^∞-function. For any r-th derivation $\eta^{(r)}$, $\eta^{(r)}(0) = 0$. Therefore, $\sum_{i=0}^\infty \frac{1}{n!} \eta^{(r)}(0) x^n$ is not equal to η on any neighborhood at 0.

2. *A real-polynomial $p(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ is a Nash function. $p(x)$ is analytic and for a non-zero real-polynomial $P(x, y) := y - p(x)$, $P(x, f(x)) = 0$.*

3. *A function $f(x) := \sqrt{1 + x^2} : \mathbb{R} \to \mathbb{R}$ is not a real-polynomial but a Nash function. $f(x)$ is analytic and for a non-zero real-polynomial $P(x, y) := y^2 - (1 + x^2)$, $P(x, f(x)) = 0$.*

4. *A function $f(x) := e^x : \mathbb{R} \to \mathbb{R}$ is analytic but not a Nash function. $f(x)$ is analytic and there does not exists a non-zero real polynomial $P(x, y)$ such that $P(x, f(x)) = 0$.*

2.2 Nash manifolds

From Introduction in [8], a subset of $\mathbb{R}^n$ is called **semialgebraic subset** if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n | f_i(x) = 0, g_j(x) > 0 \, \forall i = 1, \ldots, k, j = 1, \ldots, l\}$$

where $f_1, \ldots, f_k, g_1, \ldots, g_l$ are real polynomial functions on $\mathbb{R}^n$.

Definition 2 ([8]) *Let M be a topological space.*

1. *A C^ω-**Nash manifold** (A $\mathcal{N}^\omega$-**manifold**) is a topological space M if there exists an open finite cover $\{U_\alpha\}_\alpha$ of M, a finite family $\{V_\alpha\}_\alpha$ of open semialgebraic sets of $\mathbb{R}^n$ and homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$ such that $\phi_\beta \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta)} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a C^r Nash diffeomorphism for any α, β $(U_\alpha \cap U_\beta \neq \emptyset)$.*

2. *A C^ω **Nash function** is a C^ω-function $f : M \to \mathbb{R}$ such that $f \circ \phi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is a C^ω-Nash function for any α.*

Definition 3 1. *For a C^∞-manifold (resp. a C^ω-manifold, a $\mathcal{N}^\omega$-manifold) M, define $C^\infty(M)$ (resp. $C^\omega(M)$, $\mathcal{N}^\omega(M)$) is a set of C^∞-functions (resp. C^ω-functions , C^ω-Nash functions) on M.*

2. *$C^\infty_p(M)$ (resp. $C^\omega_p(M)$, $\mathcal{N}^\omega_p(M)$) is a set of germs of C^∞-functions (resp. C^ω-functions, C^ω-Nash functions) at p on M.*

From the definition of functions, we have a following property.

$$C^\infty(M) \supset C^\omega(M) \supset \mathcal{N}^\omega(M), \; C^\infty_p(M) \supset C^\omega_p(M) \supset \mathcal{N}^\omega_p(M).$$

We write $\mathcal{K}$ for $C^\infty, C^\omega, \mathcal{N}^\omega$, and $\mathcal{K}^\omega$ for $C^\omega, \mathcal{N}^\omega$.

3 Definitions of rings and their derivations

From Proposition 1.6.2. and 1.6.3. in [1], analytic functions on $\mathbb{R}^n$ are closed under sums, products, and partial derivations. Moreover, analytic functions are closed under compositions from Proposition 1.6.7 in [1].

From Proposition 3.1. in [7], Nash functions on $\mathbb{R}^n$ are closed under sums, products, and partial derivations.

For C^∞-functions (resp. C^ω-functions, and Nash-functions), we can define operations of sum, product, and compositions of functions as a following proposition.

Proposition 1 *$C^\infty(\mathbb{R}^n)$ (resp. $C^\omega(\mathbb{R}^n)$, $\mathcal{N}^\omega(\mathbb{R}^n)$) is closed under sums, products, and compositions, i.e.*

$$f + g, f \cdot g, h \circ (c_1, \ldots, c_m) \in \mathcal{K}(\mathbb{R}^n)$$

for any $f, g, c_1, \ldots, c_m \in \mathcal{K}(\mathbb{R}^n)$ and $h \in \mathcal{K}(\mathbb{R}^m)$. Therefore, $C^\infty(\mathbb{R}^n)$ (resp. $C^\omega(\mathbb{R}^n)$, $\mathcal{N}^\omega(\mathbb{R}^n)$) is an $\mathbb{R}$-algebra.

3.1 The definition of rings

From Proposition 1, C^∞-functions (resp. analytic-functions, Nash-functions) on $\mathbb{R}^n$ are closed under the composition $(c_1, \ldots, c_m) \mapsto f \circ (c_1, \ldots, c_m)$ by a C^∞-function (resp. an analytic-function, a Nash-function) f on $\mathbb{R}^m$. A C^∞-ring is defined in Definition 2.1. [5] by C^∞-functions. We define analytic-rings and Nash-rings as same as C^∞-rings with the following definition.

Definition 4 1. *A C^∞-**ring (differentiable ring)** (resp. C^ω-**ring (analytic-ring)**, $\mathcal{N}^\omega$-**ring (Nash-ring)**) is a set $\mathfrak{C}$ which satisfies that: for any $l \in \{0\} \cup \mathbb{N}$ and any C^∞-map (resp. C^ω-map, $\mathcal{N}^\omega$-map) $f : \mathbb{R}^l \to \mathbb{R}$ (if $l = 0$, f is a constant number of $\mathbb{R}$), there exists an operation $\Phi^{\mathfrak{C}}_f : \mathfrak{C}^l \to \mathfrak{C}$ such that*

   (a) *for any $k \in \{0\} \cup \mathbb{N}$ and any C^∞-maps (resp. C^ω-maps, $\mathcal{N}^\omega$-maps) $g : \mathbb{R}^k \to \mathbb{R}$ and $f_i : \mathbb{R}^l \to \mathbb{R}$ $(i = 1, \cdots, k)$,*

   $$\Phi^{\mathfrak{C}}_g(\Phi^{\mathfrak{C}}_{f_1}(c_1, \ldots, c_l), \ldots, \Phi^{\mathfrak{C}}_{f_k}(c_1, \ldots, c_l)) = \Phi^{\mathfrak{C}}_{g \circ (f_1, \ldots, f_k)}(c_1, \ldots, c_l) \textit{ for all } c_1, \cdots, c_l \in \mathfrak{C},$$

   (b) *for all projections $\pi_i(x_1, \ldots, x_l) = x_i (i = 1, \cdots, l)$,*

   $$\Phi^{\mathfrak{C}}_{\pi_i}(c_1, \ldots, c_l) = c_i \textit{ for all } c_1, \cdots, c_l \in \mathfrak{C}.$$

2. *A* **morphism between** $\mathcal{K}$**-rings** $\mathfrak{C},\mathfrak{D}$ *is a map* $\phi:\mathfrak{C}\to\mathfrak{D}$ *such that*

$$\Phi_f^{\mathfrak{D}}(\phi(c_1),\ldots,\phi(c_n)) = \phi\circ\Phi_f^{\mathfrak{C}}(c_1,\ldots,c_n) \text{ for all } f\in\mathcal{K}(\mathbb{R}^n), c_1,\ldots,c_n\in\mathfrak{C}.$$

We give examples of rings and homomorphisms.

Example 2 1. *Let* $\mathbb{R}$ *be a set of real numbers.* $\mathbb{R}$ *has a structure of* $\mathcal{K}$*-ring by the operation* $\Phi_f^{\mathbb{R}}:\mathbb{R}^l\to\mathbb{R}$ *for* $f\in\mathcal{K}(\mathbb{R}^l)$ *as*

$$\Phi_f^{\mathbb{R}}(r_1,\ldots,r_l) := f(r_1,\ldots,r_l) \text{ for all } r_1,\ldots,r_l\in\mathbb{R}.$$

2. *Let* M *be a* $\mathcal{K}$*-manifold.* $\mathcal{K}(M)$ *has a structure of* $\mathcal{K}$*-ring by the operation* $\Phi_f^{\mathcal{K}(M)}:\mathcal{K}(M)^l\to\mathcal{K}(M)$ *for* $f\in\mathcal{K}(\mathbb{R}^l)$ *as*

$$\Phi_f^{\mathcal{K}(M)}(c_1,\ldots,c_l) := f\circ(c_1,\ldots,c_l) \text{ for all } c_1,\ldots,c_l\in\mathcal{K}(M).$$

3. *Let* $f:M\to N$ *be a* $\mathcal{K}$*-mapping of* $\mathcal{K}$*-manifolds. Its pullback* $f^*:\mathcal{K}(N)\to\mathcal{K}(M)$ *defined as* $f^*(c):=c\circ f(c\in\mathcal{K}(N))$ *is a morphism of* $\mathcal{K}$*-rings.*

Definition 5 *Let* $\mathfrak{C}$ *be a* $\mathcal{K}$*-ring. An* $\mathbb{R}$**-point of** $\mathfrak{C}$ *is defined as a surjective homomorphism* $p:\mathfrak{C}\to\mathbb{R}$ *of* $\mathcal{K}$*-rings.*

Example 3 *Let* M *be a* $\mathcal{K}$*-manifold and* p *a point of* M. $\mathcal{K}(M)$ *has an* $\mathbb{R}$*-point* $e_p:\mathcal{K}(M)\to\mathbb{R}$ *by the operation*

$$e_p(f) = f(p) \text{ for any } f\in\mathcal{K}(M).$$

3.2 The $\mathbb{R}$-algebra structure of $\mathcal{K}$-rings

From Proposition 1, for any $\mathcal{K}$-manifold M, the $\mathcal{K}$-ring $\mathcal{K}(M)$ is the $\mathbb{R}$-algebra. As same as $\mathcal{K}(M)$, any $\mathcal{K}$-ring $\mathfrak{C}$ has the natural $\mathbb{R}$-algebra structure. From the operations Φ_f of $\mathcal{K}$-rings in Definition 4, define operations of the $\mathbb{R}$-algebra as

- the addition on $\mathfrak{C}$ by $c+c' := \Phi_{(x,y)\mapsto x+y}(c,c')$,
- the multiplication on $\mathfrak{C}$ by $c\cdot c' := \Phi_{(x,y)\mapsto xy}(c,c')$, and
- the scalar multiplication by $\lambda\in\mathbb{R}$ by $\lambda c := \Phi_{x\mapsto\lambda x}(c)$.

We see that elements 0 and 1 in $\mathfrak{C}$ are given by

- $0_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 0}(\emptyset)$ and
- $1_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 1}(\emptyset)$.

An ideal of the C^∞-ring is defined as an ideal of the commutative $\mathbb{R}$-algebra ([5]). Then, an ideal of the analytic-ring (resp. Nash-ring) is defined as an ideal of the commutative $\mathbb{R}$-algebra as same as C^∞-rings.

We have Hadamard's Lemma for C^∞-functions, C^ω-functions, and Nash-functions as a following corollary.

Corollary 1 *For any* $\mathcal{K}$*-functions* $f\in\mathcal{K}(\mathbb{R}^n)$*, there exists* n*-*$\mathcal{K}$*-functions* $g_1,\ldots,g_n\in\mathcal{K}(\mathbb{R}^{2n})$ *such that*

$$f(x+y)-f(x) = \sum_{i=1}^{n} y_i g_i(x,y) \text{ for all } x=(x_1,\ldots,x_n), y=(y_1,\ldots,y_n)\in\mathbb{R}^n.$$

Therefore, we define a quotent $\mathcal{K}$-ring by an ideal of the $\mathcal{K}$-ring as same as Definition 2.7. in [5]. Let $\mathfrak{C}$ be a $\mathcal{K}$-ring and $I\subset\mathfrak{C}$ be an ideal as an $\mathbb{R}$-module. From Corollary 1, for any $f\in\mathcal{K}(\mathbb{R}^n)$, there exists $g_1,\ldots,g_n\in\mathcal{K}(\mathbb{R}^{2n})$ such that

$$\Phi_f^{\mathfrak{C}}(c_1+i_1,\ldots,c_n+i_n) - \Phi_f^{\mathfrak{C}}(c_1,\ldots,c_n) = \sum_{j=1}^{n} i_j\Phi_{g_i}^{\mathfrak{C}}(c_1,\ldots,c_n,i_1,\ldots,i_n)$$

for any $c_1,\ldots,c_n\in\mathfrak{C}$ and $i_1,\ldots,i_n\in I$. We can define a quotient $\mathcal{K}$-ring $\mathfrak{C}/I$ as

$$\Phi_f^{\mathfrak{C}/I}(c_1+I,\ldots,c_n+I) := \Phi_f^{\mathfrak{C}}(c_1,\ldots,c_n)+I \text{ for any } f\in\mathcal{K}(\mathbb{R}^n), c_1+I,\ldots,c_n+I\in\mathfrak{C}/I.$$

3.3 Localizations and local rings

Definition 6 (Joyce [5] for C^∞-rings) *Let $\mathfrak{C}$ be a $\mathcal{K}$-ring and p an $\mathbb{R}$-point of $\mathfrak{C}$.*

1. *We call a $\mathcal{K}$-ring $\mathfrak{C}_p$ with following properties a* ***localization*** *of $\mathfrak{C}$ at p.*
   (a) *There exists a unique morphism $\pi_p : \mathfrak{C} \to \mathfrak{C}_p$ such that*
   $$\pi_p(s) \text{ is invertible in } \mathfrak{C}_p \text{ for all } s \in p^{-1}(\mathbb{R}\backslash\{0\}). \tag{1}$$
   (b) *If there exists a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ which satisfies (1), there exists a unique morphism $\phi_p : \mathfrak{C}_p \to \mathfrak{D}$ such that $\phi_p \circ \pi_p = \phi$.*
2. *A $\mathcal{K}$-ring $\mathfrak{C}$ is called a* **$\mathcal{K}$-local ring** *if $\mathfrak{C}$ has a unique maximal ideal $m_{\mathfrak{C}}$ which satisfies $\mathfrak{C}/m_{\mathfrak{C}} \cong \mathbb{R}$.*

Any localization $\mathfrak{C}_p$ of $\mathcal{K}$-ring $\mathfrak{C}$ at any $\mathbb{R}$-point p is a $\mathcal{K}$-local ring with a maximal ideal $m_p \subset \mathfrak{C}_p$.

For a $\mathcal{K}$-manifold M and its point p with an $\mathbb{R}$-point e_p defined as $e_p(f) = f(p)$, there exists a localization $\mathcal{K}(M)_p := \mathcal{K}(M)_{e_p}$ of $\mathcal{K}(M)$ at e_p with a homomorphism $\pi_p : \mathcal{K}(M) \to \mathcal{K}(M)_p$ defined as $\pi_p(f) = f/1$. The set $\mathcal{K}_p(M)$ of germs of $\mathcal{K}$-functions on M at p has a homomorphism $\phi_p : \mathcal{K}(M) \to \mathcal{K}_p(M)$ defined as $\phi_p(f) := [f, M]_p$.

For any $f \in \mathcal{K}(M)$ with $f(p) \neq 0$, $\phi_p(f) := [f, M]_p$ has an invertible element $[\frac{1}{f}, f^{-1}(\mathbb{R}\backslash\{0\})]_p$. Then, there exists a unique homomorphism $\iota_p : \mathcal{K}(M)_p \to \mathcal{K}_p(M)$ such that $\iota_p \circ \pi_p = \phi_p$.

3.4 $\mathbb{R}$-derivations and $\mathcal{K}$-derivations

For [3], $\mathbb{R}$-derivations on a C^∞-ring are defined as $\mathbb{R}$-derivations of the $\mathbb{R}$-algebra. C^∞-derivations on a C^∞-ring are defined in [5]. We define C^ω-derivations (analytic-derivations) and $\mathcal{N}^\omega$-derivations (Nash-derivations) as same as C^∞-derivations with the following definition.

Definition 7 (Joyce [5] for C^∞-rings) *Suppose $\mathfrak{C}$ is a $\mathcal{K}$-ring, and $\mathfrak{M}$ a $\mathfrak{C}$-module.*

1. *An* **$\mathbb{R}$-derivation** *is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ with*
   $$d(c_1c_2) = c_2d(c_1) + c_1d(c_2) \text{ for any } c_1, c_2 \in \mathfrak{C}.$$
2. *A* **$\mathcal{K}$-derivation** *is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ with*
   $$d(\Phi_f(c_1, \ldots, c_n)) = \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot d(c_i) \text{ for any } n \in \mathbb{N}, f \in \mathcal{K}(\mathbb{R}^n), c_1, \ldots, c_n \in \mathfrak{C}.$$
3. *Let $d : \mathfrak{C} \to \mathfrak{M}$ be a K-derivation ($K = \mathbb{R}, \mathcal{K}$). We call a pair $(\mathfrak{M}, d)$ a* **K-cotangent module** *for $\mathfrak{C}$ if for any K-derivation $d' : \mathfrak{C} \to \mathfrak{M}'$ there exists a unique morphism $\phi : \mathfrak{M} \to \mathfrak{M}'$ of $\mathfrak{C}$-modules such that $\phi \circ d = d'$. Then we write $(\Omega_{\mathfrak{C},K}, d_{\mathfrak{C},K})$ for the K-cotangent module for $\mathfrak{C}$.*

We have that any $\mathcal{K}$-derivation is an $\mathbb{R}$-derivation since there exists a function $f(x_1, x_2) = x_1x_2$ with $\frac{\partial f}{\partial x_2} = x_1, \frac{\partial f}{\partial x_1} = x_2$ which is a product of $\mathcal{K}$-rings.

For a Nash-function f on an open set $\mathbb{R}^n$, a partial derivation of f is also the Nash-function ([7]). Therefore, we have a following example of derivations as partial derivations.

Example 4 *Let U be an open set of $\mathbb{R}^n$. $C^\infty(U)$ (resp. $C^\omega(U)$, $\mathcal{N}^\omega(U)$) is closed under partial derivations $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, i.e.*

$$\frac{\partial f}{\partial x_i} \in \mathcal{K}(U) \text{ for any } i = 1, \ldots, n, f \in \mathcal{K}(U).$$

Therefore, for $c_1, \ldots, c_n \in \mathcal{K}(U)$, $\sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} : \mathcal{K}(U) \to \mathcal{K}(U)$ is a $\mathcal{K}$-derivation.

We show examples of C^∞-derivations for C^∞-mainfolds.

Example 5 *Let M be a C^∞-manifold and $\Gamma(T^*M)$ be the set of C^∞-sections to the cotangent bundle T^*M on M.*

1. *For any $f \in C^\infty(M)$, define a C^∞-section $df : M \to T^*M$ by*
$$df(v) := v(f) \text{ for any } x \in M \text{ and } v \in T_xM.$$
Define an $\mathbb{R}$-derivation $d : C^\infty(M) \to \Gamma(T^*M)$ as $d(f) := df$.

2. *Let $V : M \to TM$ be a C^∞-vector field of M as $V_x \in T_xM$ for $x \in M$. Define $V(f) \in C^\infty(M)$ by*
$$(V(f))(x) := V_x(f).$$
We regard $V : C^\infty(M) \to C^\infty(M)$ as an $\mathbb{R}$-derivation.

4 Differentiable rings

4.1 Germs of C^∞-functions

Lemma 1 *Let M be a C^∞-manifold, p a point of M and U an open neighborhood at p of M.*

1. *There exists $\eta \in C^\infty(M)$ and an open neighborhood V at p of U such that*
$$\eta(x) = \begin{cases} 1 & (x \in V) \\ 0 & (x \in M \backslash U) \end{cases}.$$

2. *For any $f \in C^\infty(U)$, there exists $g \in C^\infty(M)$ such that $f|_V \equiv g|_V$.*

4.2 The localizations for C^∞-rings

Let M be a C^∞-manifold. Suppose that $p \in M$ is a point and U is an open neighborhood of M at p. We can define a morphism $\pi_p^U : C^\infty(U) \to C_p^\infty(M)$ of C^∞-rings as $\pi_p^U(f) := [f, U]_p$.

From Lemma 1, we have a following property about germs of C^∞-functions on a C^∞-manifold.

Corollary 2 $\pi_p^U : C^\infty(U) \to C_p^\infty(M)$ *is surjective.*

We have defined the morphism $\iota_p : C^\infty(M)_p \to C_p^\infty(M)$. From the following corollary, ι_p is isomorphism and $(C_p^\infty(M), \pi_p^M)$ is regarded as a localization of $C^\infty(M)$ at p.

Corollary 3 (Joyce [5]) *For a C^∞-ring $C^\infty(M)$ and its $\mathbb{R}$-point $e_p : C^\infty(M) \to \mathbb{R}$ by a point $p \in M$, we have the isomorphism $C^\infty(M)_{e_p} \cong C_p^\infty(M)$.*

4.3 Derivations of k-jet determined C^∞-rings

$C^\infty(M)$ is embedded to $\prod_{p\in M} \mathbb{R}$ by $f \mapsto \{f(p)\}_{p\in M}$. Then $C^\infty(M)$ is point determined in Definition 4.1. [6]. We define k-jet determined as the generalization of point determined.

Definition 8 (Yamashita [9]) *Let $\mathfrak{C}$ be a C^∞-ring and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. For an $\mathbb{R}$-point p of $\mathfrak{C}$, define a homomorphism $j_p^k : \mathfrak{C} \to \mathfrak{C}_p/{m_p}^{k+1}$ by $j_p^k(c) := \pi_p(c) + m_p^{k+1}$ for $c \in \mathfrak{C}$ (If $k = \infty$, we mean ${m_p}^{k+1}$ by ${m_p}^\infty := \cap_{k\in\mathbb{N}} {m_p}^k$). Define $j^k : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$ as $j^k := (j_p^k)_{p:\mathfrak{C}\to\mathbb{R}}$.*

*$\mathfrak{C}$ is a k-**jet determined** C^∞-**ring** if j^k is injective.*

Theorem 1 (Yamashita [9]) *Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings and $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. Suppose that $\mathfrak{D}$ is k-jet determined.*

Then any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ over ϕ is a C^∞-derivation.

Example 6 *Let M be a C^∞-manifold. There exists an injection $j^0 : C^\infty(M) \to \prod_{p\in M} \mathbb{R}$ defined as $j^0(f) := \{f(p)\}_{p\in M}$. Then, $C^\infty(M)$ be a 0-jet determined C^∞-ring.*

Therefore, from Theorem 1, for a smooth mapping $f : M \to N$ of C^∞-manifolds, any $\mathbb{R}$-derivation $V : C^\infty(N) \to C^\infty(M)$ along $f^* : C^\infty(N) \to C^\infty(M)$ is C^∞-derivation.

The condition of k-jet determined is need to Theorem 1. From D. Joyce [5] Remark 5.5., we have an example that all $\mathbb{R}$-derivation are not C^∞-derivations.

Example 7 (Joyce [5]) *Let $\mathfrak{C}$ be a C^∞-ring $C^\infty(\mathbb{R})$.*

For the $\mathbb{R}$-cotangent module $(\Omega_{\mathfrak{C},\mathbb{R}}, d_{\mathfrak{C},\mathbb{R}})$, $d_{\mathfrak{C},\mathbb{R}} : \mathfrak{C} \to \Omega_{\mathfrak{C},\mathbb{R}}$ is an $\mathbb{R}$-derivation but not a C^∞-derivation.

$\Omega_{\mathfrak{C},C^\infty}$ is a finitely generated $\mathfrak{C}$-module generated by $d(x)$. $\Omega_{\mathfrak{C},\mathbb{R}}$ is not a finitely generated generated $\mathfrak{C}$-module.

In fact, for the exponential $e^x \in C^\infty(\mathbb{R})$, $e^x d_{\mathfrak{C},\mathbb{R}}(x) - d_{\mathfrak{C},\mathbb{R}}(e^x) \neq 0$ in $\Omega_{\mathfrak{C},\mathbb{R}}$.

5 Analytic-rings

5.1 Germs of analytic functions

From Lemma 1, for a C^∞-manifold M and a point p, there exists a C^∞-function $\eta \in C^\infty(M)$ and open neighborhoods $V \subset U$ of M at p such that $\eta|_V \equiv 1$ and $\eta|_{M\setminus\overline{U}} \equiv 0$.

For a connected C^ω-manifold (resp. $\mathcal{N}^\omega$-manifold) M, a germ $[f,U]_p$ of C^ω-functions which is extendable to M have a unique function $g \in C^\infty(M)$ whose germ is $[f,U]_p$ for following lemmas.

Lemma 2 *Let M be a connected C^ω-manifold and f be a real-C^ω-function on M. $f = 0$ on M if and only if there exists a non-empty open subset $U \subset M$ such that $f|_U \equiv 0$.*

Lemma 3 *Let f be a real-C^ω-function on $\mathbb{R}^n$. $f = 0$ on $\mathbb{R}^n$ if and only if there exists a point $p \in \mathbb{R}^n$ such that $\frac{\partial^\alpha f}{\partial x^\alpha}(p) = 0$ for any α.*

5.2 Germs of analytic rings and Nash rings

Let M be a C^ω-manifold(resp. $\mathcal{N}^\omega$-manifold). Suppose that $x \in M$ is a point and $i : V \hookrightarrow U$ is an inclusion of open connected subsets in M. We can define morphisms of $\mathcal{K}^\omega$-rings as

$$\rho_{UV} : \mathcal{K}^\omega(U) \ni f \mapsto f|_V \in \mathcal{K}^\omega(V),$$
$$\pi_p^U : \mathcal{K}^\omega(U) \ni f \mapsto [f,U]_p \in \mathcal{K}_p^\omega(M).$$

From Lemma 2 and Lemma 3, we have a following corollary.

Corollary 4 1. *ρ_{UV} and π_p^U are injective.*

2. *For a maximal ideal $m_p = \{f \in \mathcal{K}^\omega(M) | f(p) = 0\}$ of $\mathcal{K}^\omega(M)$, $m_p^\infty = \cap_{i=1}^\infty m_p^i = 0$.*

We have defined the morphism $\iota_p : C^\omega(M)_p \to C_p^\omega(M)$. ι_p is not isomorphism. For example, take an analytic function $f(x) := \frac{1}{1-x}$ on $(-1,1)$ and a point $0 \in \mathbb{R}$. $f(x) = \sum_{i=0}^\infty x^n$ on $(-1,1)$. We can't take $g \in C^\omega(\mathbb{R})$ such that $[g,\mathbb{R}]_0 = [f,(-1.1)]_0$. Therefore, $\iota_0 : C^\omega(\mathbb{R})_0 \to C_0^\omega(\mathbb{R})$ is not surjective, moreover not isomorphism.

6 Nash-rings

6.1 The condition of Nash functions by Kähler-differential spaces

Suppose that $U \subset \mathbb{R}^n$ be an open connected semialgebraic subset.

For Nash-functions on $\mathbb{R}^n$, we have a following theorem for Nash-derivations of Nash-rings.

Theorem 2 (Ishikawa-Yamashita [4]) *A real analytic function $f \in C^\omega(U)$ is a Nash-function if and only if there exists a Nash function $g \in \mathcal{N}^\omega(U) (g \neq 0)$ such that*

$$g\Big(df - \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\Big) = 0$$

in $\Omega_{C^\omega(U),\mathbb{R}}$ and $\Omega_{C^\infty(U),\mathbb{R}}$

From Theorem 2 about Nash-functions and Nash-derivations, we have a following property.

Theorem 3 *Suppose that $\mathfrak{A}$ is a Nash-ring and $\mathfrak{M}$ be an $\mathfrak{A}$-module with an $\mathbb{R}$-derivation $V : \mathfrak{A} \to \mathfrak{M}$. For any $f \in \mathcal{N}^{\omega}(\mathbb{R}^n)$ and $c_1, \dots, c_n \in \mathfrak{A}$,*

$$V(\Phi_f(c_1, \dots, c_n)) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \dots, c_n) V(c_i)$$

is a nilpotent element in $\mathfrak{M}$.

6.2 Ideals of Nash rings

For C^∞-rings, all finitely generated C^∞-rings are not finitely presented C^∞-rings, i.e. C^∞-rings of the form $C^\infty(\mathbb{R}^n)/\langle f_1, \dots, f_k \rangle_{C^\infty(\mathbb{R}^n)}$ by $f_1, \dots, f_k \in C^\infty(\mathbb{R}^n)$.

From Corollary 1.5.5.in p41 [8] and Theorem 8.7.15 in [2], we have a following proposition.

Proposition 2 (Shiota [8]) *Suppose that M be an affine $\mathcal{N}^{\omega}$-manifold, such that there exists an C^ω Nash embedding $i : M \hookrightarrow \mathbb{R}^m$. $\mathcal{N}^{\omega}(M)$ is a Noether ring, i.e. for any ideal $I \subset \mathcal{N}^{\omega}(M)$, there exists finite functions $g_1, \dots, g_k \in I$ such that $I = \langle g_1, \dots, g_k \rangle_{\mathcal{N}^{\omega}(M)}$.*

Therefore, any finite generated $\mathcal{N}^{\omega}$-ring is finitely presented.

6.3 The localizations of Nash-rings

For a sheaf $\mathcal{O}_X$ on a topological space X, a sequence $\cdots \mathcal{F}^{i-1} \to \mathcal{F}^{i} \to \mathcal{F}^{i+1} \to \cdots$ of $\mathcal{O}_X$ is exact if and only if a sequence $\cdots \mathcal{F}_p^{i-1} \to \mathcal{F}_p^{i} \to \mathcal{F}_p^{i+1} \to \cdots$ of $\mathcal{O}_{X,p}$ is exact for any $p \in X$ ([3]).

Proposition 3 (Shiota [8]) *For a Nash-ring $\mathcal{N}^{\omega}(\mathbb{R}^n)$ and its $\mathbb{R}$-point $e_p : \mathcal{N}^{\omega}(\mathbb{R}^n) \to \mathbb{R}$, $\mathcal{N}_p^{\omega}(\mathbb{R}^n)$ is faithfully flat on $\mathcal{N}^{\omega}(\mathbb{R}^n)_{e_p}$, i.e. any sequence of $\mathcal{N}^{\omega}(\mathbb{R}^n)_{e_p}$*

$$\cdots \to \mathfrak{N}_{i-1} \to \mathfrak{N}_i \to \mathfrak{N}_{i+1} \to \cdots,$$

is exact if and only if the sequence of $\mathcal{N}_p^{\omega}(\mathbb{R}^n)$

$$\cdots \to \mathfrak{N}_{i-1} \otimes_{\mathcal{N}^{\omega}(\mathbb{R}^n)_{m_p}} \mathcal{N}_p^{\omega}(\mathbb{R}^n) \to \mathfrak{N}_i \otimes_{\mathcal{N}^{\omega}(\mathbb{R}^n)_{m_p}} \mathcal{N}_p^{\omega}(\mathbb{R}^n) \to \mathfrak{N}_{i+1} \otimes_{\mathcal{N}^{\omega}(\mathbb{R}^n)_{m_p}} \mathcal{N}_p^{\omega}(\mathbb{R}^n) \to \cdots$$

is exact.

For a C^∞-ring $C^\infty(M)$, above Proposition satisfies since $C_p^\infty(M) \doteqdot C^\infty(M)_{e_p}$ for any point $p \in M$

A Analytic-ringed spaces and Nash-ringed spaces

A C^∞-ringed space is defined in [5]. We define C^ω-ringed spaces (resp. Nash-ringed spaces) and a functor Spec as same as C^∞-rings with the following definitions.

A.1 The definition of analytic-ringed spaces and Nash-ringed spaces

Definition 9

1. *A $\mathcal{K}^{\omega}$-ringed space $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of $\mathcal{K}^{\omega}$-rings on X.*

2. *A morphism $\underline{f} = (f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $\mathcal{K}^{\omega}$-ringed spaces is a continuous map $f : X \to Y$ and a morphism $f^{\#} : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ of sheaves of $\mathcal{K}^{\omega}$-rings on Y.*

3. *A local $\mathcal{K}^{\omega}$-ringed space $\underline{X} = (X, \mathcal{O}_X)$ is a $\mathcal{K}^{\omega}$-ringed space for which $\mathcal{O}_{X,x}$ are $\mathcal{K}^{\omega}$-local rings for all $x \in X$.*

A.2 The examples of analytic-ringed spaces and Nash-ringed spaces

Example 8 *Let M be a $\mathcal{K}^\omega$-manifold.*

1. *For an open set $U \subset M$, define $\mathcal{O}_M(U)$ as a set of functions $s : U \to \coprod_{x \in U} \mathcal{K}^\omega(M)_{e_p}$ such that for any point $x \in U$, there exists a open neighborhood $V \subset U$ at p and elements $c, d \in \mathcal{K}^\omega(M)(d(p) \neq 0)$ which satisfy $s(q) = \pi_q(c)\pi_q(d)^{-1}$ for any point $q \in U$.*
2. *Define $\mathcal{O}'_M$ as $\mathcal{O}'_M(U) := \mathcal{K}^\omega(U)$ for any open set $U \subset M$.*

Then, $(M, \mathcal{O}_M)$ and $(M, \mathcal{O}'_M)$ are $\mathcal{K}^\omega$-ringed spaces.

A.3 The definition of Spec

Definition 10 (Joyce [5]) 1. *For a C^∞-ring $\mathfrak{C}$, define a C^∞-ringed space$\underline{X_{\mathfrak{C}}}$ as followings.*

   (a) Define a topological space $X_{\mathfrak{C}}$ as followings by C^∞-ring $\mathfrak{C}$.
   - *Define a set $X_{\mathfrak{C}} := \{x : \mathfrak{C} \to \mathbb{R} | x \text{ is a } \mathbb{R}\text{-point of } \mathfrak{C}.\}$.*
   - *For each $c \in \mathfrak{C}$, define $c_* : X_{\mathfrak{C}} \ni x \mapsto x(c) \in \mathbb{R}$.*
   - *Set a topology of $X_{\mathfrak{C}}$ as a smallest topology $\mathcal{T}_{\mathfrak{C}}$ such that c_* is continuous for all $c \in \mathfrak{C}$.*

   (b) For an open subset $U \subset X_{\mathfrak{C}}$, define $\mathcal{O}_{X_{\mathfrak{C}}}(U)$ as a set of functions $s : U \to \coprod_{x \in U} \mathfrak{C}_x$ with following properties
   - *For each $x \in U$, $s(x) \in \mathfrak{C}_x$ is satisfied.*
   - *U is covered by open set V with*
     for some $c, d \in \mathfrak{C}(\forall x \in V, \pi_x(d) \neq 0)$, $\pi_x(c)\pi_x(d)^{-1} = s(x)(\forall x \in V)$ is satisfied.

2. *Therefore define the following C^∞-ringed space*

$$\mathit{Spec}\, \mathfrak{C} := (X_{\mathfrak{C}}, \mathcal{O}_{X_{\mathfrak{C}}}).$$

References

[1] Ralph P Boas and Harold P Boas. *A primer of real functions*, Vol. 13. Cambridge University Press, 1996.

[2] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Géométrie algébrique réelle (Real algebraic geometry)*, Vol. 12. Springer Science & Business Media, 1987.

[3] Robin Hartshorne. *Algebraic Geometry, volume 52 of Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[4] Go-o Ishikawa and Tatsuya Yamashita. Leibniz complexity of Nash functions on differentiations. *arXiv preprint arXiv:1509.08261*, 2015.

[5] Dominic Joyce. Algebraic geometry over C^∞-rings. *arXiv preprint arXiv:1001.0023v6*, 2015.

[6] Ieke Moerdijk and Gonzalo E Reyes. *Models for smooth infinitesimal analysis*. Springer Science & Business Media, 2013.

[7] Rodolphe Ramanakoraisina. Complexité des fonctions de Nash. *Communications in Algebra*, Vol. 17, No. 6, pp. 1395–1406, jan 1989.

[8] Masahiro Shiota. *Nash manifolds, Lectures Notes in Math., 1269*. Springer-Verlag, 1987.

[9] Tatsuya Yamashita. Derivations on a C^∞-ring. *Communications in Algebra*, Vol. 44, No. 11, pp. 4811–4822, jun 2016.

Department of Mathematics
Hokkaido University
Hokkaido 065-0006
JAPAN
E-mail address: tatsuya-y@math.sci.hokudai.ac.jp

北海道大学大学院理学院数学専攻 山下　達也